

# General Relativity: The Riemann-Christoffel tensor

Gonzalo E. Reyes  
Université de Montréal

6 May 2006

## 0.1 Curvature of a connection

Following Lavendhomme [1] we define the curvature of the connection  $\nabla$  as the map  $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$  defined by

$$R_{XY}Z = (C \circ C^D - C \circ C^D \circ \Sigma)(X \star Y \star Z)$$

where  $X, Y, Z \in \mathcal{X}(M)$ ,  $C : M^{D \times D} \rightarrow M^D$  is the connection map of  $\nabla$  and  $(X \star Y \star Z)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_2} \circ X_{d_1}$

In the following proposition,  $\sum_0$  denotes the "cyclic sum" defined by  $\sum_0 \phi(X_1, X_2, X_3) = \phi(X_1, X_2, X_3) + \phi(X_2, X_3, X_1) + \phi(X_3, X_1, X_2)$

### Proposition 0.1

1.  $R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$
2.  $R_{X_2 X_1} X_3 = -R_{X_1 X_2} X_3$
3. Assume that  $\nabla$  is symmetric. Then  $\sum_0 R_{X_1 X_2} X_3 = 0$ . (First Bianchi identity).

*Proof:*

1. Already done (cf. "Curvaturetorsion.pdf" Proposition 0.14, page 12)

2. Immediate from the first and  $\nabla_{-X}Y = -\nabla_X Y$ . (cf. "Covariantderivation.pdf" Theorem 3, page 3)
3. Since  $\nabla$  is symmetric,  $\nabla_X Y - \nabla_Y X = [X, Y]$  (cf. "Curvaturetorsion.pdf" Proposition 0.15, page 16). Compute the LHS using this relation repeatedly

$$\begin{aligned}
\sum_0 R_{X_1 X_2} X_3 &= \sum_0 (\nabla_{X_1} \nabla_{X_2} X_3 - \nabla_{X_2} \nabla_{X_1} X_3 - \nabla_{[X_1, X_2]} X_3) \\
&= \sum_0 \nabla_{X_1} [X_2, X_3] - \sum_0 \nabla_{[X_1, X_2]} X_3 \\
&= \sum_0 (\nabla_{X_1} [X_2, X_3] - \nabla_{[X_2, X_3]} X_1) \\
&= \sum_0 [X_1, [X_2, X_3]] \\
&= 0 \text{ (Jacobi's identity)}
\end{aligned}$$

## 0.2 Curvature of a connection when $M = R^n$ .

We start with the relation

$$R_{\partial/\partial x^i \partial/\partial x^j} \partial/\partial x^k = \nabla_{\partial/\partial x^i} (\nabla_{\partial/\partial x^j} \partial/\partial x^k) - \nabla_{\partial/\partial x^j} (\nabla_{\partial/\partial x^i} \partial/\partial x^k)$$

which follows from 0.1 and  $[\partial/\partial x^i, \partial/\partial x^j] = 0$ .

We now use the relation

$$\nabla_{\partial/\partial x^i} \partial/\partial x^j = \sum_k \Gamma_{ij}^k \partial/\partial x^k$$

(cf, "Covariantderivation.pdf" page 4). Therefore,

$$\begin{aligned}
(1) \quad \nabla_{\partial/\partial x^i} (\sum_l \Gamma_{jk}^l \partial/\partial x^l) &= \sum_l \nabla_{\partial/\partial x^i} (\Gamma_{jk}^l \partial/\partial x^l) \\
&= \sum_l (\partial \Gamma_{jk}^l / \partial x^i \partial/\partial x^l + \Gamma_{jk}^l \nabla_{\partial/\partial x^i} \partial/\partial x^l) \\
&= \sum_l \partial \Gamma_{jk}^l / \partial x^i \partial/\partial x^l + \sum_l \Gamma_{jk}^l \nabla_{\partial/\partial x^i} \partial/\partial x^l
\end{aligned}$$

Similarly,

$$(2) \quad \nabla_{\partial/\partial x^j} (\sum_l \Gamma_{ik}^l \partial/\partial x^l) = \sum_l \partial \Gamma_{ik}^l / \partial x^j \partial/\partial x^l + \sum_l \Gamma_{ik}^l \nabla_{\partial/\partial x^j} \partial/\partial x^l$$

Letting  $R_{ij}^k = R_{\partial/\partial x^i \partial/\partial x^j} \partial/\partial x^k$  we have

$$R_{ij}^k = (1) - (2)$$

F and its  $l^{th}$  component is readily computed to be

$$(R_{ij}^k)_l = \partial \Gamma_{jk}^l / \partial x^i - \partial \Gamma_{ik}^l / \partial x^j + \sum_{\alpha} (\Gamma_{jk}^{\alpha} \Gamma_{i\alpha}^l - \Gamma_{ik}^{\alpha} \Gamma_{j\alpha}^l)$$

This coincides with what Spivak (vol II, page 189) and Einstein ("The meaning of Relativity", page 74, fifth edition) call  $R_{kij}^l$ , but with  $\Gamma_{ji}^l$  instead of our  $\Gamma_{ij}^l$ . Notice, however, that this difference is spurious. Since  $\nabla$  is symmetric,  $\Gamma_{ij}^k = \Gamma_{ji}^k$  as the following argument shows:

$$\nabla(t_1, t_2)(d_1, d_2) = x + d_1 e_i + d_2 e_j + d_1 d_2 \nabla_4((x, e_i), (x, e_j))$$

Therefore,

$$\nabla(t_2, t_1)(d_2, d_1) = x + d_2 e_j + d_1 e_i + d_1 d_2 \nabla_4((x, e_j), (x, e_i))$$

By definition of  $\nabla$  being symmetric,  $\nabla(t_1, t_2)(d_1, d_2) = \nabla(t_2, t_1)(d_2, d_1)$  and this implies that  $\Gamma_{ij}^k = -(\nabla_4(x, e_j), (x, e_i))_k = -(\nabla_4(x, e_i), (x, e_j))_k = \Gamma_{ji}^k$ .

### 0.3 The Riemann-Christoffel tensor

Let  $M$  be microlinear with a symmetric connection  $\nabla$ . In this section we define the tensor of the title in terms of the curvature  $R$  of  $\nabla$ .

We define the *Riemann-Christoffel tensor* to be the map  $R : M^D \times_M M^D \times_M M^D \rightarrow M^D$  given by the prescription

$$R(t_1, t_2, t_3) = (R_{X_1 X_2 X_3})_m$$

where  $t_1, t_2, t_3 \in M_m$  and  $(X_i)_m = t_i$ .

For this definition to make sense, we must show the following

**Lemma 0.2** *Assume that  $X_m = X'_m, Y_m = Y'_m, Z_m = Z'_m$ . Then*

$$(R_{XY}Z)_m = (R_{X'Y'}Z')_m$$

*Proof:* Notice first that  $(R_{XY}Z)_m = (R_{X'Y}Z)_m$ , i.e., a letter may be replaced by the corresponding primed letter if it occurs in first position. This follows from the definition of the curvature, namely

$$R_{XY}Z = (C \circ C^D - C \circ C^D \circ \Sigma)(Z \star Y \star X)_m$$

by observing that  $(Z \star Y \star X)_m = (Z \star Y \star X')_m$ . In fact,

$$\begin{aligned}
(Z \star Y \star X)_m(d_1, d_2, d_3) &= (Z_{d_3} \circ Y_{d_2} \circ X_{d_1})_m \\
&= (Z_{d_3} \circ Y_{d_2})(X_m(d_1)) \\
&= (Z_{d_3} \circ Y_{d_2})(X'_m(d_1)) \\
&= (Z_{d_3} \circ Y_{d_2} \circ X'_{d_1})_m \\
&= (Z \star Y \star X')_m(d_1, d_2, d_3)
\end{aligned}$$

Using the previous proposition,

$$\begin{aligned}
R_{X'Y'Z'} &= -R_{Y'Z'}X' - R_{Z'X'}Y' \\
&= R_{Z'Y'}X' - R_{Z'X'}Y' \\
&= R_{ZY'}X' - R_{ZX'}Y' \\
&= -R_{Y'Z}X' - R_{ZX'}Y' \\
&= R_{X'Y'}Z \\
&= R_{XY}Z
\end{aligned}$$

**Proposition 0.3** *The Riemann-Christoffel tensor has the following properties*

$$\begin{aligned}
R(t_2, t_1, t_3) &= -R(t_2, t_1, t_3) \\
R(t_1, t_2, t_3) + R(t_2, t_3, t_1) + R(t_3, t_1, t_2) &= 0 \\
R(\lambda t_1, t_2, t_3) &= \lambda R(t_1, t_2, t_3) \\
R(t_1, \lambda t_2, t_3) &= \lambda R(t_1, t_2, t_3) \\
R(t_1, t_2, \lambda t_3) &= \lambda R(t_1, t_2, t_3)
\end{aligned}$$

*Proof:* Immediate from the corresponding ones for  $R_{XY}Z$  in "curvature-tor-sion.pdf" page 17.

**Conjecture:**  $R(t_1, t_2, t_3) = R_{t_1 t_2} t_3$ , where the RHS refers to the Riemann-Christoffel tensor defined à la Cartan (cf. mo/re and ???)

For finite dimensional manifolds, this follows from the Proposition in page 236 of Moerdijk/Reyes. However, no proof of this proposition is given beyond the comment that it follows from "a horrible computation".

## 0.4 Contraction of a tensor

PUT THIS IN CLASSICAL CONTEXT ONLY! If  $R^l_{ijk}$  is the curvature tensor, we define  $R_{ij} = \sum_l R^l_{ijl}$  to be the Ricci tensor. The operation of the passage from the first tensor to the second is called "contraction".

From a more conceptual point of view, the Riemann-Christoffel curvature tensor is given by a map  $R : M^D \times_M M^D \times_M M^D \rightarrow M^D$ . The contraction is a new map  $Ric : M^D \times_M M^D \rightarrow M^D$  defined as follows: if  $u, v \in M_x$ , let  $\Psi(u, v) : M_x \rightarrow M_x$  be the  $R$ -linear map defined by  $\Psi(u, v)(w) = R_{uw}v$ . Then  $Ric(u, v) = trace(\Psi(u, v))$ .

To connect this invariant definition of  $Ric$  with the classical expression of the Ricci tensor we take any basis of  $M_x$  to represent  $\Psi(u, v)$  as a matrix. Then  $Ric(u, v)$  is the trace of this matrix. Take, for instance the basis  $\{\partial/\partial x^i|_x\}_i$ . Then the matrix of  $\Psi(u, v)$  relative to this basis is  $(\omega^i(\Psi(u, v)(\partial/\partial x^j|_x)))_{ij}$ , where  $\{\omega^i\}_i$  is the dual basis. Then its trace is  $Ric(u, v) = \sum_i \omega^i(R_{u(\partial/\partial x^i|_x)}v)$ . In coordinates, if  $u = \partial/\partial x^j|_x$  and  $v = \partial/\partial x^i|_x$ , then the Riemann-Christoffel tensor is  $R^l_{ijk}$  and the Ricci tensor is indeed  $\sum_l R^l_{ijl} = R_{ij}$ , i.e., its contraction.

**Proposition 0.4** *If  $\nabla$  is symmetric, then*

$$Ric(u, v) = Ric(v, u)$$

*Proof:* From proposition ??  $R(u, \partial/\partial x_i|_x) + R(\partial/\partial x_i|_x, v, u) + R(v, u, \partial/\partial x_i|_x) = 0$ . Thus  $R(u, \partial/\partial x_i|_x, v) - R(v, \partial/\partial x_i|_x, u) = R(u, v, \partial/\partial x_i|_x)$  which in turn implies that

$$Ric(u, v) - Ric(v, u) = \sum_i \omega^i R(u, v, \partial/\partial x_i|_x)$$

We are reduced to prove

**Claim:**  $\sum_i \omega^i R(u, v, \partial/\partial x_i|_x) = 0$

Another way of posing the problem: let  $\phi : M_x \rightarrow M_x$  be the linear transformation defined by  $\phi(w) = R(u, v, w)$ . Then its matrix is skew symmetric. In fact ???

REST OF THE PROOF???

I cannot prove this proposition, but something much weaker:

**Lemma 0.5** *For every  $x \in R$ ,  $x^2 - x + 1 > 0$ .*

*Proof:* LATER!

**Lemma 0.6** *Let  $v_1, \dots, v_n \in R$ . There exists  $v_0 > 0$  such that for every  $i$   $v_0 + v_i$  and  $v_0 - v_i$  are units.*

*Proof:* Take  $v_0 = 1 + \sum_i v_i^2$  and use previous lemma.

**Corollary 0.7** *Let  $\phi : R^n \times R^n \rightarrow R$  be multilinear such that  $\phi(u, u) = 0$  whenever  $u \neq 0$ . Then  $\phi$  vanishes on the diagonal.*

*Proof:* Let  $v \in R^n$ . Then  $v = (v_1, \dots, v_n)$ . By the previous lemma there is  $v_0$  such that both  $v_0 + v_i$  and  $v_0 - v_i$  are units. Let  $u = (v_0, \dots, v_0)$ . Hence  $u$ ,  $u + v$  and  $u - v$  are non-zero. Thus

$$\begin{aligned} 0 &= \phi(u + v, u + v) = \phi(u, u) + \phi(u, v) + \phi(v, u) + \phi(v, v) \\ 0 &= \phi(u - v, u - v) = \phi(u, u) + \phi(u, -v) + \phi(-v, u) + \phi(-v, -v) \end{aligned}$$

Adding these two equations and using multilinearity we obtain the desired result. CHECK!

### Einstein' vacuum field equations

$$Ric(u, u) = 0 \text{ for every } u \in M_x$$

To conclude  $Ric = 0$  it is enough to show that  $Ric$  is symmetric. I don't know whether this can be proved in this general context. However, by defining the symmetrization of  $Ric$ ,  $Ric^{sym}(u, v) = 1/2(R(u, v) + R(v, u))$  we can write these equations as

$$Ric^{sym} = 0.$$

(In fact, develop  $Ric(u + v, u + v) = Ric(u, u) + 2Ric^{sym}(u, v) + Ric(v, v)$ )

## References

- [1] R.Lavendomme, *Basic Concepts of Synthetic Differential Geometry*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1996
- [2] I.Moerdijk and G.E.Reyes